

Stability of Anomalous Floquet Edge Unitaries

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It has recently been shown that periodically driven noninteracting systems may exhibit anomalous chiral edge modes, despite hosting bands with trivial topology. We obtain many-body versions of such drives, corresponding to class A, which exhibit anomalous charge transport at the edge and which reduce to their noninteracting counterparts for systems of single particles. We extend these drives to more general systems of bosons, fermions and spins, and characterise their anomalous action at the edge of an open system by an effective edge unitary. These are found to be robust to all local perturbations and may be classified by a pair of coprime integers.

Introduction. Time-dependent systems can support a host of novel phenomena that are impossible to realise with a static Hamiltonian. These include topological adiabatic cycles [1–6], Floquet analogues of topological insulators [7–19]; novel examples of driven, interacting, symmetry-protected topological phases (SPTs) [20–24]; and phases which exhibit an analogue of spontaneous symmetry breaking in the time domain, dubbed time crystals [25–28]. In addition to being of theoretical interest, much progress has been made towards realising Floquet systems in the laboratory [29–34].

Many of the unusual Floquet phases mentioned above are distinguished by their anomalous edge behaviour: While a periodic drive may behave trivially in a closed system, its action at the boundary of an open system can be non-trivial. This kind of holography has been studied successfully by a number of different groups, many of which use many-body localisation (MBL) as part of the description. Our approach will instead be to introduce specific model drives (which we call unitary loops) that may be interpreted without reference to MBL, although MBL would undoubtedly be important for any physical realisation of these phases (see Supplemental Material).

The unitary evolution by a Hamiltonian $H(t)$ is $U(T) = \mathcal{T} \exp\left(-i \int_0^T H(t) dt\right)$, where \mathcal{T} is the time-ordering operator and T is the period of evolution. We define a unitary loop to be an evolution of this form which, in a closed system, satisfies $U(T) = \mathbb{I}$. In the corresponding open system, however, $U(T)$ will not necessarily be proportional to the identity. The component of $U(T)$ that acts in the vicinity of the boundary, which we call the effective edge unitary, characterises the anomalous edge action of the unitary evolution. It is useful to study unitary loops, since any generic unitary evolution whose endpoint in a closed system takes the form $U(T) = \exp(-iH_F)$, with static Floquet Hamiltonian H_F , may be decomposed into a unitary loop followed by an evolution due to H_F (this includes all MBL evolutions). This decomposition property is proved in the Supplemental Material, where it is also proved that the effective edge unitary is well defined for any unitary loop. Our approach will be to study the effective edge unitaries

for anomalous interacting SPT drives.

Two fundamental examples of SPT phases are those of class D, protected by particle-hole symmetry, and class A, protected by $U(1)$ charge conservation. In the time-dependent case, a one-dimensional (1D) class D system has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ classification, exhibiting two phases that are inherently dynamical [8, 10, 16]. Several groups have shown that this classification scheme persists in the presence of interactions [21, 23, 24].

In two dimensions (2D), class A corresponds to the integer quantum Hall effect (IQHE) [35]. IQHE phases are classified by an integer, which is also well-defined in the presence of interactions [36]. An interesting example of a nontrivial driven system belonging to class A was given in Ref. 9, which shows IQHE-like chiral edge modes at the boundary of a 2D lattice, even though the bulk band has Chern number zero. This model has recently been shown to be stable to disorder [17].

In the first section of this article we set out to examine the effect of interactions on this drive by constructing its many-body analogues. In the single-particle model in Ref. 9, the particle statistics were unimportant, but in our case, we find that the behaviour of the drive depends subtly on whether we are dealing with bosons or fermions. We specifically consider the anomalous edge action, and find that charge conservation protects this from being destroyed by any local, charge-conserving, 1D unitary evolution. The structure of these particle-based models motivates a more general set of exchange models, which we introduce in the context of spin systems. We find that these also exhibit a robust edge action, and we provide a classification scheme for anomalous behaviour of this kind.

Interacting Class A Drive. We first introduce an interacting drive in class A that reduces to the drive of Ref. 9 for a single-particle system. For simplicity, we initially work with hardcore bosons, but will later extend our model to fermions and spins. As in Ref. 9, the aim will be to generate circulating particle currents that cancel out in the bulk, but which lead to propagation along the edges of an open system. Since the action of a unitary of this kind is the identity for a closed system, this

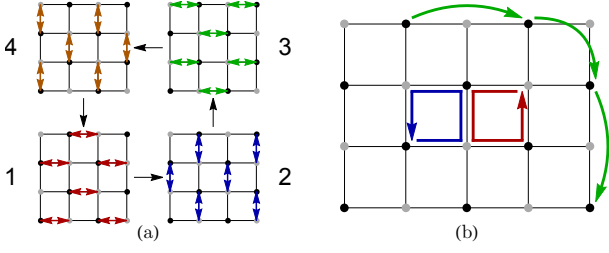


Figure 1. (a) Four steps in the anomalous Floquet drives considered in the main text, based on the model of Ref. 9. For the hardcore boson model, each step of the drive interchanges the occupation numbers on the pairs of states on the highlighted set of bonds. (b) Representation of the action of the complete unitary drive on some single-site components of a generic state. On-site states in the bulk are mapped onto themselves, while some states on the edge are translated.

is an example of a nontrivial unitary loop [19, 24].

The drive in Ref. 9 consists of four principal steps, each of which generates hopping across a different set of neighbouring bonds, as shown in Fig. 1(a). After the complete drive, a particle in the bulk returns to its initial position, but a particle initially located at certain positions on the edge is transported along the boundary, represented pictorially in Fig. 1(b). For hardcore bosons, the unitary that generates a hop between two sites is

$$U_{\mathbf{r}\mathbf{r}'}^B = 1 + b_{\mathbf{r}}^\dagger b_{\mathbf{r}'} + b_{\mathbf{r}}^\dagger b_{\mathbf{r}'} - b_{\mathbf{r}}^\dagger b_{\mathbf{r}} - b_{\mathbf{r}'}^\dagger b_{\mathbf{r}'} + 2b_{\mathbf{r}}^\dagger b_{\mathbf{r}'} b_{\mathbf{r}}^\dagger b_{\mathbf{r}}, \quad (1)$$

where $b_{\mathbf{r}}^\dagger$ creates a boson on site \mathbf{r} and satisfies $[b_{\mathbf{r}}, b_{\mathbf{r}'}^\dagger] = \delta_{\mathbf{r},\mathbf{r}'}$ and $(b_{\mathbf{r}}^\dagger)^2 = 0$. It may be verified that this operator is unitary and that it acts on a general two-site state as

$$U_{\mathbf{r}\mathbf{r}'}^B \left(b_{\mathbf{r}}^\dagger \right)^{n_{\mathbf{r}}} \left(b_{\mathbf{r}'}^\dagger \right)^{n_{\mathbf{r}'}} |0\rangle = \left(b_{\mathbf{r}}^\dagger \right)^{n_{\mathbf{r}'}} \left(b_{\mathbf{r}'}^\dagger \right)^{n_{\mathbf{r}}} |0\rangle, \quad (2)$$

with $n_{\mathbf{r}}, n_{\mathbf{r}'} \in \{0, 1\}$. Labelling the two sublattices as A and B (filled and open circles, respectively, in Fig. 1), and setting the intersite spacing to one, each step of the unitary drive may be written $U_j^B = \prod_{\mathbf{r} \in A} U_{\mathbf{r}, \mathbf{r}+\mathbf{b}_j}^B$ with $\mathbf{b}_1 = (1, 0)$, $\mathbf{b}_2 = (0, -1)$, $\mathbf{b}_3 = (-1, 0)$ and $\mathbf{b}_4 = (0, 1)$. The complete unitary drive is then $U^B = U_4^B U_3^B U_2^B U_1^B$, which can be written as the product of evolutions by four local Hamiltonians.

Within each step of the drive, the two-site operators $U_{\mathbf{r},\mathbf{r}'}^B$ act on disjoint pairs of sites and commute. By tracking the position of a particular particle across all steps of the unitary, it can be verified that the action of the complete drive has the form shown pictorially in Fig. 1(b). On a many-body product state, the unitary acts as a permutation of the particle occupation numbers. Since the unitary acts identically on any product state, the permutation is also well defined for superposition states.

Since the drive only affects particles at the boundary, the effective edge unitary of the drive may be read off directly from the complete time evolution operator. We write a generic many-body product state as

$|n_1, n_2, \dots, n_{N_x N_y}\rangle$, where n_j gives the boson occupation number on site j and $N_x N_y$ is the dimension of the system. The unitary drive maps product states onto product states through the relation $|n'_1, n'_2, \dots, n'_{N_x N_y}\rangle = U_{\{n\}}^{B, \{n'\}} |n_1, n_2, \dots, n_{N_x N_y}\rangle$. From the discussion above, the matrix elements are

$$U_{\{n\}}^{B, \{n'\}} = \prod_{j \in \text{bulk}} \delta_{n_j, n'_j} \prod_{j \in \text{edge}} \delta_{n_j, n'_{j+1}}, \quad (3)$$

where we assume that the sites at the edge are indexed appropriately [37]. The effective edge unitary, U_{eff}^B , is characterised by the matrix elements of the second factor.

It is natural to ask whether this many-body generalisation applies also to fermionic systems. We define fermionic unitary operators, $U_{\mathbf{r}\mathbf{r}'}^F$, U_j^F and U^F by replacing $b_{\mathbf{r}}^\dagger$ with $f_{\mathbf{r}}^\dagger$ in the bosonic definitions above. The operators $f_{\mathbf{r}}^\dagger$ satisfy $\{f_{\mathbf{r}}, f_{\mathbf{r}'}^\dagger\} = \delta_{\mathbf{r},\mathbf{r}'}$ and have the occupation number-exchanging property

$$U_{\mathbf{r}\mathbf{r}'}^F \left(f_{\mathbf{r}}^\dagger \right)^{n_{\mathbf{r}}} \left(f_{\mathbf{r}'}^\dagger \right)^{n_{\mathbf{r}'}} |0\rangle = \left(f_{\mathbf{r}}^\dagger \right)^{n_{\mathbf{r}'}} \left(f_{\mathbf{r}'}^\dagger \right)^{n_{\mathbf{r}}} |0\rangle, \quad (4)$$

with $n_{\mathbf{r}}, n_{\mathbf{r}'} \in \{0, 1\}$. In this case, the presence of the vacuum state $|0\rangle$ is important. For a many-body Slater determinant, anticommuting the relevant fermion operators so that they are adjacent to the vacuum will introduce an overall sign, which depends on the occupation numbers of all other lattice sites. In this way, the fermionic drive U^F acting on a closed system may return a Slater determinant state to itself, or to minus itself; in an open system, the unitary translates particles at the edge only up to a sign. The fermionic matrix elements are related to their bosonic counterparts through $U_{\{n\}}^{F, \{n'\}} = (-1)^s U_{\{n\}}^{B, \{n'\}}$, where s is an integer that depends non-locally on $\{n'\}$ and $\{n\}$. For a superposition state, the unitary may introduce different signs for different components.

Nevertheless, the fermionic Floquet drive U^F has many interesting properties, and also exhibits anomalous edge behaviour. Any charge distribution at the edge of a generic many-body state will be translated around the boundary in a chiral manner. Furthermore, if the drive is run twice (which we call the ‘doubled fermion drive’), then its action in the bulk is exactly the identity, since the sign factors square to one. This is reminiscent of fermionic Hamiltonians that avoid the fermion sign problem. In this case, the bulk and edge behaviour can be disentangled and an effective edge unitary can be defined (see Supplemental Material).

The drives described above have been constructed to give the desired edge behaviour, and one might ask whether these fine-tuned examples are truly representative of a finite region of parameter space. We now argue that this is the case, and that the anomalous action is stable to local unitaries at the edge. We consider the bosonic version of the drive here and briefly address the fermionic drive further below.

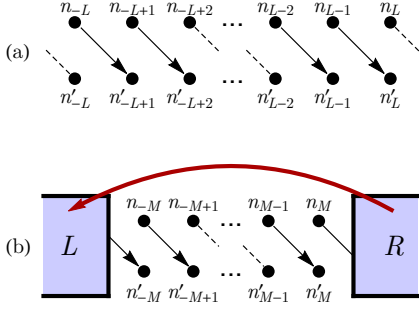


Figure 2. (a) Permutation of boson occupation numbers at the edge of a many-body product state under the anomalous Floquet drive. (b) Action of the putative open 1D unitary \tilde{U}_{op}^B . The regions marked L and R are regions of the chain within a distance $v_{LR}T$ of the cut. In the bulk, the permutation action is unchanged, and so charge must be transferred from R to L . See main text for details.

To proceed, we consider the action of the effective unitary restricted to the 1D edge, which, without loss of generality, we take to have length $2L$. Acting on a product state, we find $U_{\text{eff}}^B |n_{-L}, n_{-L+1}, \dots, n_L\rangle = |n_L, n_{-L}, \dots, n_{L-1}\rangle$, shown pictorially in Fig. 2(a).

We will now assume that this unitary may be generated by a local, 1D charge-conserving Hamiltonian $H(t)$ that acts for a finite time T at the edge, and we will show that this leads to a contradiction [38]. Since $H(t)$ is local, the complete drive will have a maximum Lieb-Robinson velocity for the speed of information propagation, v_{LR} [39].

We can obtain an open version of the drive, \tilde{U}_{op}^B , by cutting open the 1D edge and consistently excluding all terms that connect sites across the cut. Due to the Lieb-Robinson bound, this cannot have an effect on the unitary evolution outside of a region $\Delta j \approx v_{LR}T$ near the left and right ends. In this way, for a long chain with $L \gg v_{LR}T$, we expect the unitary far from the cut to have the same action as before.

Now, assume that in a large length of the chain from site $j = -M$ to site $j = M$, the action of the unitary is unaffected by the cut, as shown in Fig. 2(b). Since the charge in the bulk is transported uniformly by one lattice site through the action of the unitary, it follows from charge conservation that the charge initially in sites $\{-L, -L+1, \dots, -M-1\} \cup \{M, M+1, \dots, L\}$ must equal the final charge in sites $\{-L, -L+1, \dots, -M\} \cup \{M+1, M+2, \dots, L\}$. However, the available space for charge on the right is reduced by this evolution, while the available space for charge on the left is increased. The only way that total charge can be conserved for *any* initial charge configuration is if particles are transferred from the right edge to the left edge to address any imbalance. This distance can be made arbitrarily large by increasing the system size, which shows that in general \tilde{U}_{op}^B must be nonlocal (or that T must be infinite). We conclude that

the anomalous action of U_{eff}^B cannot arise as result of a local, 1D Hamiltonian $H(t)$ acting for a finite time.

For the fermionic models, this argument shows that there is no local 1D unitary which brings back the action of the open system to that of the closed system. Furthermore, for the doubled fermion drive, the above bosonic argument can be straightforwardly applied to demonstrate the anomalous nature of the edge unitary.

Exchange Models. The models described above may be generalised straightforwardly to spin models, or indeed any system where the on-site Hilbert spaces are equivalent. Instead of particle hops, the building blocks are now pairwise exchanges of local states. The exchange version of Eq. (1) is

$$U_{\mathbf{r}, \mathbf{r}'}^{\leftrightarrow} = \sum_{\alpha \neq \beta} |\mathbf{r}, \beta\rangle \otimes |\mathbf{r}', \alpha\rangle \langle \mathbf{r}, \alpha| \otimes \langle \mathbf{r}', \beta| + \delta_{\alpha\beta} \mathbb{I}_{\mathbf{r}\mathbf{r}'}, \quad (5)$$

where $\alpha, \beta \in \mathcal{H}_{\mathbf{r}}$ take values in the on-site Hilbert space and the identity acts if the on-site states are equal. In the above, $|\mathbf{r}, \alpha\rangle$ indicates that the state at site \mathbf{r} is α .

The anomalous Floquet evolution consists of four parts and has a similar form to before. In each step, the tensor product of the exchange operation is taken over one of the four sets of neighbouring bonds shown in Fig. 1, $U_j^{\leftrightarrow} = \bigotimes_{\mathbf{r} \in A} U_{\mathbf{r}, \mathbf{r}+\mathbf{b}_j}^{\leftrightarrow}$, with the complete drive given by $U^{\leftrightarrow} = U_4^{\leftrightarrow} U_3^{\leftrightarrow} U_2^{\leftrightarrow} U_1^{\leftrightarrow}$. Each step consists of a product of local, commuting terms, and so can be generated by a local Hamiltonian. The complete action of the drive may again be represented as in Fig. 1(b), where now, instead of particle hops, the arrows indicate the trajectory of a particular on-site state through the lattice. Acting on a product state, U^{\leftrightarrow} permutes the on-site states through a cyclic permutation at the edge, an action that is also well-defined for superposition states. This may be encapsulated in an effective edge unitary, $U_{\text{eff}}^{\leftrightarrow}$.

A natural setting for this type of anomalous drive is a lattice of spins. If the on-site Hilbert space corresponds to \mathbb{Z}_2 , then the model maps formally onto the hardcore boson model given previously. More general spin models may be mapped onto bosonic models that allow a different (but finite) number of particles per site. The Hamiltonians that generate these drives conserve total boson number, and their edge action can be shown to be anomalous using the arguments given previously.

Stability of Edge Unitaries. We now allow for the possibility of perturbations that do not conserve charge. In these cases we can appeal to more general information theoretic ideas to show that the effective edge action of the Floquet drive is still anomalous. Roughly speaking, the anomalous edge drives have a chiral flow of information (and not just charge) which we will show cannot occur through a local 1D unitary evolution. As before, we will assume the converse is true and identify a contradiction. We again begin by assuming that $U_{\text{eff}}^{\leftrightarrow}$ may be generated by a local Hamiltonian $H(t)$, and so there is a maximum velocity v_{LR} at which information can flow.

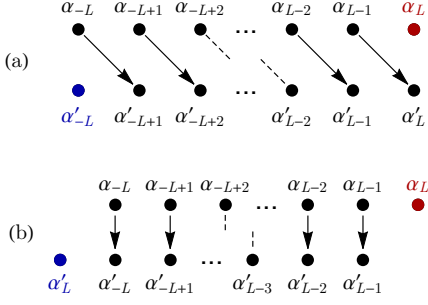


Figure 3. (a) Permutation of on-site states under the action of $\tilde{U}_{\text{op}}^{\leftrightarrow}$, assuming the 1D chain is cut between sites L and $-L$. (b) Relabelling of lattice sites so that the permutation is diagonal away from the cut.

We cut open the chain to obtain the putative open system unitary $\tilde{U}_{\text{op}}^{\leftrightarrow}$, which should reproduce the permutation action in the bulk of the chain away from the cut.

For simplicity, we will assume that the edge region (of size $v_{LR}T$) consists of a single site either side of the cut (for the more general case, see the Supplemental Material). With this setup, the action of $\tilde{U}_{\text{op}}^{\leftrightarrow}$ on a many-body state is to translate the on-site states to the right by one lattice site, as shown in Fig. 3(a). The unitary $\tilde{U}_{\text{op}}^{\leftrightarrow}$ maps product states onto product states through the matrix elements $|\alpha'_{-L}, \alpha'_{-L+1}, \dots, \alpha'_L\rangle = U_{\{\alpha\}}^{\{\alpha'\}} |\alpha_{-L}, \alpha_{-L+1}, \dots, \alpha_L\rangle$, where $|\{\alpha\}\rangle$ and $|\{\alpha'\}\rangle$ are initial and final states, respectively. These matrix elements have the form

$$U_{\{\alpha\}}^{\{\alpha'\}} = \prod_{j=-L}^{L-1} \delta_{\alpha_j, \alpha'_{j+1}} f(\{\alpha\}, \{\alpha'\}), \quad (6)$$

where the final factor describes the relation between states α_L and α'_{-L} .

We now relabel the site indices in the final state through $j \rightarrow j' = j + 1$ in the range $-L \leq j' \leq L$, which makes the permutation diagonal (see Fig. 3(b)). In this new basis, the matrix elements of the unitary are

$$U_{\{\alpha\}}^{\{\alpha'\}} = \delta_{\alpha_{\mathbf{v}}, \alpha'_{\mathbf{v}}} f(\alpha_{\mathbf{v}} = \alpha'_{\mathbf{v}}, \alpha_L, \alpha'_L), \quad (7)$$

where we have used the shorthand notation \mathbf{v} to represent the sites from $-L$ to $L-1$. From the unitarity of $U_{\{\alpha\}}^{\{\alpha'\}}$, it may be shown that the factor $f(\alpha_{\mathbf{v}}, \alpha_L, \alpha'_L)$ is also unitary. The unitary evolution $\tilde{U}_{\text{op}}^{\leftrightarrow}$ therefore relates state α'_L to α_L through an (unspecified) unitary operation.

Now, the unitary evolution as a whole preserves information. The diagonal factor in $U_{\{\alpha\}}^{\{\alpha'\}}$ shows that information in the bulk is translated, while the remaining factor $f(\alpha_{\mathbf{v}}, \alpha_L, \alpha'_L)$ shows that the information in state α_L is transferred to state α'_L unitarily. However, in the original basis, these states were separated by an arbitrarily large distance, and transferring information across this distance in a finite time would violate the Lieb-Robinson

bound. In this way, the anomalous edge action of a general exchange model is robust, and cannot be created or destroyed by a 1D unitary drive of the form $\tilde{U}_{\text{op}}^{\leftrightarrow}$.

We can construct more general drives by stacking together several systems (and thereby acting on a tensor product Hilbert space) or by running several drives in sequence. Drives generated in this way are not necessarily independent, as we now show.

The action of a general exchange drive can be characterised by a permutation of the form $|\alpha_1, \alpha_2, \dots, \alpha_L\rangle \rightarrow |\alpha_{L-p+1}, \alpha_{L-p+2}, \dots, \alpha_{L-p}\rangle$, which moves each state on the edge to the right by p lattice sites. If the on-site Hilbert space has dimension k , then we write this right-moving permutation as $R(p, k)$. Left-moving permutations may similarly be written $L(p, k)$.

We note that running the drive $R(p, k)$ q times is equivalent to running the drive $R(qp, k)$ once. Secondly, we note that by grouping together the first p lattice sites into a single effective site, the drive $R(p, k)$ is equivalent to the drive $R(1, k^p)$ [40]. This regrouping of sites is equivalent to the stacking together of p drives on different Hilbert spaces with dimension k . Stacking more general drives leads to the equivalence

$$R(p, k) \otimes R(p', k') \equiv R(1, k^p (k')^{p'}). \quad (8)$$

In this way, any right-moving drive is equivalent to a drive $R(1, n)$, where n is a positive integer. Using the same methods as above, it is straightforward to show that drives corresponding to different n are inequivalent (i.e. $R(1, n)$ cannot be obtained from $R(1, n')$ through a local 1D unitary evolution for $n \neq n'$).

In the Supplemental Material, we show that by also including left-moving permutations, a generic permutation can be brought into the form $L(1, n') \otimes R(1, n)$, where n and n' are coprime integers. A general exchange drive may therefore be characterised by a pair of integers, describing left and right-moving components of the permutation. A trivial drive can be reduced to the form $n = 1, n' = 1$. Again using the methods above, all of these drives can be shown to be inequivalent.

In general, an effective edge unitary, U_{eff} , will not correspond to a pure exchange drive. From our results it follows that any effective edge unitary that is equivalent to an exchange effective edge unitary can be characterised by coprime integers n, n' . We conjecture that this classification is also complete, i.e. that every effective edge unitary belongs to one of these equivalence classes.

Conclusions. In summary, we have presented a many-body version of the anomalous Floquet drive of Ref. 9, which is applicable to both bosonic and fermionic systems. The action of the drive leads to the robust chiral propagation of charge at the boundary of an open system. Anomalous edge behaviour arises more generally in exchange models, where spin states, for example, are swapped between Hilbert spaces on neighbouring sites. Finite bounds on the propagation of information mean

that classes of anomalous edge behaviour are stable to local perturbations. We showed that exchange drives may be uniquely characterised (up to equivalence) by a pair of coprime integers, and we conjectured that all effective edge unitaries are equivalent to one of these exchange drives. As well as introducing several new interacting Floquet topological phases, our work raises a number of interesting questions which we hope will encourage further theoretical and experimental efforts. For instance, it would be interesting to study the interacting analogues and stability of other single-particle Floquet topological insulators.

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Note — In the process of preparing this manuscript we became aware of Ref. [41], which considers chiral Floquet phases in the context of MBL systems. Our results, while framed in a different setting, seem consistent with this work.

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Supplemental Material

UNITARY LOOPS AND MANY-BODY LOCALISATION

As stated in the main text, many studies of driven topological systems rely on many-body localisation (MBL), a phenomenon in which weak disorder breaks ergodicity and permits a description of many-body states in terms of local, conserved quantities known as l-bits (see Ref. 42 for a review). In an MBL system, the notion of a quantum phase may be applied to the complete spectrum of eigenstates, rather than to a particular gapped ground state, and becomes well-defined even at finite temperature [43–48]. Moreover, since a driving Hamiltonian transfers energy to the system, MBL is believed to be required to prevent heating to infinite temperature [20, 49–57], and may also be needed for the observation of topological signatures such as persistent edge modes [43, 46].

In this article, we discuss explicit unitary loop models that do not require MBL for their anomalous edge features to be interpreted. Nevertheless, any physically reasonable realisation of these models is likely to require MBL for the reasons given above. More generally, any unitary evolution whose final expression in a closed system takes the form $U(T) = \exp(-iH_F T)$, with some well-defined static Floquet Hamiltonian H_F , may be decomposed into a unitary loop followed by an evolution by H_F . This includes all MBL unitary evolutions. To see this, consider the homotopy Hamiltonian

$$H'(t, s) = \begin{cases} H\left(\frac{t}{1-3s/4}\right) & 0 \leq t \leq T(1-3s/4) \\ -4H_F s & T(1-3s/4) \leq t \leq T(1-s/2) \\ +2H_F s & T(1-s/2) \leq t \leq T, \end{cases} \quad (\text{S1})$$

which preserves the endpoint of the evolution as s is varied. When $s = 0$, this is the original time-dependent Hamiltonian $H(t)$, but when $s = 1$, the first half of the evolution is a unitary loop and the second half of the evolution is a scaled evolution by the Hamiltonian H_F . In this way, any unitary evolution of the form $U(T) = \exp(-iH_F T)$ can be rewritten as a unitary loop followed by a constant evolution. The unitary loop component captures the inherently dynamical physics that is not described by the Floquet Hamiltonian. See Refs [19, 24] for further discussion of unitary loops.

PROOF OF THE EXISTENCE OF AN EFFECTIVE EDGE UNITARY

In this section, we prove that a unitary evolution that behaves as the identity in a closed system always yields a well-defined effective edge unitary in the open system. The effective edge unitary will be shown to be localised near the boundary of the open system. We will implicitly work in two dimensions, assuming the closed system is a torus and the open system has been cut open to give a cylinder.

We write the time-dependent Hamiltonian that generates the unitary loop as $H(t)$. Since this must be local, we can consistently define the Hamiltonian of the open system by excluding the terms that connect parts of the system on either side of the cut. Specifically, we write

$$H_{\text{op}}(t) = H_{\text{cl}}(t) + H_e(t), \quad (\text{S2})$$

where $H_{\text{cl}}(t)$ is the Hamiltonian of the closed system, $H_{\text{op}}(t)$ is the Hamiltonian of the open system, and $H_e(t)$ includes (the negative of) the edge terms that connect sites either side of the cut.

We define unitary evolutions of the closed and open systems over the time interval $[t_1, t_2]$ in the usual manner,

$$U_{\text{cl/op}}(t_2, t_1) = \mathcal{T} \exp \left[-i \int_{t_1}^{t_2} H_{\text{cl/op}}(t') dt' \right], \quad (\text{S3})$$

where \mathcal{T} is the time-ordering operator. We now define a modified edge Hamiltonian through

$$H'_e(t) = U_{\text{cl}}(T, t) H_e(t) [U_{\text{cl}}(T, t)]^{-1}. \quad (\text{S4})$$

Since all parts of the Hamiltonian are local, the modified Hamiltonian $H'_e(t)$ is also local: conjugation with $U_{\text{cl}}(T, t)$ can at most increase the range of Hamiltonian $H_e(t)$ by a length $v_{LR}(T-t)$, where v_{LR} is the (maximum) Lieb-Robinson velocity of $H_{\text{cl}}(t)$.

Defining the infinitesimal time interval $\Delta t \equiv \lim_{N \rightarrow \infty} T/N$, we write the open system unitary as the infinite product

$$U_{\text{op}}(T) = e^{-iH_{\text{cl}}(T)\Delta t} e^{-iH_e(T)\Delta t} e^{-iH_{\text{cl}}(T-\Delta t)\Delta t} e^{-iH_e(T-\Delta t)\Delta t} \dots e^{-iH_{\text{cl}}(\Delta t)\Delta t} e^{-iH_e(\Delta t)\Delta t}. \quad (\text{S5})$$

Then, recalling Eq. (S4), we further rewrite the open system unitary in terms of the modified edge Hamiltonian as

$$U_{\text{op}}(T) = e^{-iH'_e(T)\Delta t} e^{-iH'_e(T-\Delta t)\Delta t} \dots e^{-iH'_e(\Delta t)\Delta t} U_{\text{cl}}(T), \quad (\text{S6})$$

where in the final factor $U_{\text{cl}}(T)$, we have already taken the infinitesimal limit to sum the terms involving $H_{\text{cl}}(t)$. However, by definition, the closed system unitary operator is just the identity, and so we identify

$$U_{\text{op}}(T) = \mathcal{T} \exp \left[-i \int_0^T H'_e(t') dt' \right] \equiv U_{\text{eff}}. \quad (\text{S7})$$

The Hamiltonian $H'_e(t)$ is localised near the boundary of the open system, and so U_{eff} is an effective edge operator that also acts only in the vicinity of the boundary. Other than locality, we have not assumed any properties of the Hamiltonian $H(t)$, and so the effective edge unitary is well defined for bosonic, fermionic and spin systems.

Finally, $U_{\text{op}}(T)$ cannot entangle the two ends of the cylinder since they are separated by an arbitrarily large distance. The effective unitary must therefore factorise as $U_{\text{eff}} = U_{\text{eff}}^L U_{\text{eff}}^R$, where L and R label the two ends of the cylinder.

INFORMATION FLOW WITH A FINITE EDGE REGION

In the main text, we showed that the anomalous edge action of an exchange model was robust to local evolutions at the edge, by cutting open the 1D edge and assuming only a single site each side of the cut was affected. More generally, there will be a finite edge region affecting $M \approx v_{LR}T$ sites either side of the cut. In this case, since the information in the bulk is translated, the information initially in sites $\{-L, -L+1, \dots, -M-1\} \cup \{M, M+1, \dots, L\}$ must be related to the information that ends up in sites $\{-L, -L+1, \dots, -M\} \cup \{M+1, M+2, \dots, L\}$. (This is similar to the charge conservation argument given in the main text). From the unitarity of the complete evolution, the information in these edge regions must be related through a unitary transformation $U_{LR}^{L'R'}$, where L, R and L', R' label the initial and final edge regions, respectively. However, this unitary cannot factorise into the local form $U_L^{L'} U_R^{R'}$ because $\dim(R') < \dim(R)$ and $\dim(L') > \dim(L)$. The only way that information can be preserved is if some is transferred from R to L' , which would violate the Lieb-Robinson bound.

EXAMPLE OF A UNITARY LOOP WITH TRIVIAL EFFECTIVE EDGE UNITARY

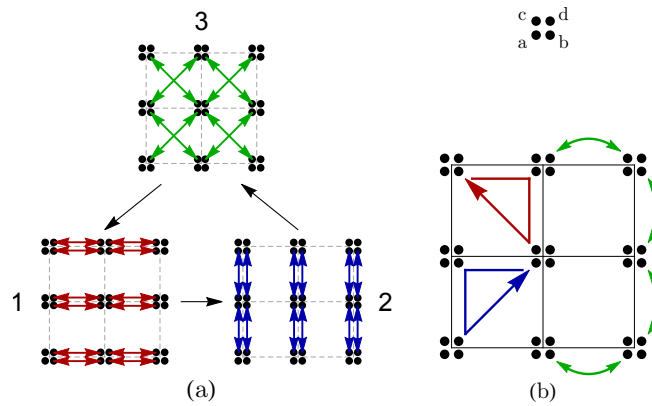


Figure S1. (a) Example of a unitary loop that generates a trivial effective edge unitary that is not the identity. On-site degrees of freedom have been drawn spatially separated for clarity. (b) Labelling of the on-site degrees of freedom and representation of the action of this drive. See main text for details.

In this section, we give an explicit example of a unitary loop that gives an effective edge unitary which, while not equal to the identity, is in the trivial class of permutations. This means that the effective edge unitary may be reduced to the identity through the action of local unitaries.

We consider a square lattice with four sublattice degrees of freedom per site, as shown in Fig. S1. Writing the wavefunction corresponding to state α on site \mathbf{r} and sublattice a as $|\mathbf{r}, a, \alpha\rangle$, a general exchange operation is written

$$U_{\mathbf{r},a;\mathbf{r}',a'}^{\leftrightarrow} = \sum_{\alpha \neq \beta} |\mathbf{r}, a, \beta\rangle \otimes |\mathbf{r}', a', \alpha\rangle \langle \mathbf{r}, a, \alpha| \otimes \langle \mathbf{r}', a', \beta| + \delta_{\alpha\beta} \mathbb{I}_{\mathbf{r},a;\mathbf{r}',a'}. \quad (\text{S8})$$

The drive then consists of the three steps,

$$\begin{aligned} U_1 &= \prod_{\mathbf{r}} U_{\mathbf{r},b;\mathbf{r}+\hat{\mathbf{x}},a}^{\leftrightarrow} U_{\mathbf{r},d;\mathbf{r}+\hat{\mathbf{x}},c}^{\leftrightarrow} \\ U_2 &= \prod_{\mathbf{r}} U_{\mathbf{r},c;\mathbf{r}+\hat{\mathbf{y}},a}^{\leftrightarrow} U_{\mathbf{r},d;\mathbf{r}+\hat{\mathbf{y}},b}^{\leftrightarrow} \\ U_3 &= \prod_{\mathbf{r}} U_{\mathbf{r},d;\mathbf{r}+\hat{\mathbf{x}}+\hat{\mathbf{y}},a}^{\leftrightarrow} U_{\mathbf{r},b;\mathbf{r}+\hat{\mathbf{x}}-\hat{\mathbf{y}},c}^{\leftrightarrow}. \end{aligned} \quad (\text{S9})$$

The overall action of this drive is represented pictorially in Fig. S1(b): in the bulk, states return to their initial locations after three steps, while on the edge, some neighbouring states are swapped. The effective edge unitary is therefore a commuting product of pairwise exchanges on the edge, which may be generated by a local Hamiltonian $H(t)$. A Hamiltonian that undoes these local exchanges will reduce the effective edge unitary to the identity.

FURTHER DETAILS ON CLASSIFICATION OF PERMUTATIONS

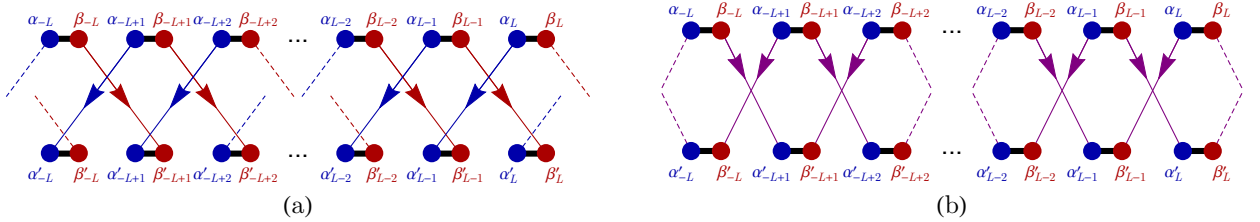


Figure S2. (a) Representation of the composite drive $L(1, k) \otimes R(1, k)$, with two copies of the Hilbert space $\mathcal{H}_{\mathbf{r}}$ per site. States from one on-site Hilbert space are translated to the left by the action of the drive, while states from the other on-site Hilbert space are translated to the right. (b) Representation of the composite drive $L(1, k) \otimes R(1, k)$ after acting with a product of local unitaries that interchanges the two on-site Hilbert spaces in the final state. The action of the drive is now a product of pairwise swaps. See main text for details.

In the main text, we showed that any combination of right-moving permutations can be cast into the form $R(1, n)$, where n is a positive integer. We now also consider left-moving permutations. It is clear that $R(p, k)$ followed by $L(q, k)$, with $q < r$ is equivalent to $R(p - q, k)$. It can also be shown that the stacking $L(p, k) \otimes R(p, k)$ is trivial. To argue this, we will show that $L(1, k) \otimes R(1, k)$ is trivial, from which the more general result follows.

The product drive $L(1, k) \otimes R(1, k)$ acts on the on-site tensor product Hilbert space $\mathcal{H}_{\mathbf{r}} \otimes \mathcal{H}_{\mathbf{r}}$, represented by the blue and red circles in Fig. S2. Under the action of the drive, states from one Hilbert space move one site to the right, while states on the other Hilbert space move one site to the left. The characteristic permutation nature of the drive is preserved under the action of local unitaries, however, which may mix together the on-site Hilbert spaces and act on clusters of nearby sites.

In this case, we imagine acting after the drive with a product of local unitaries, each of which interchanges the two states on a given site. If the two on-site Hilbert spaces are labelled a and b , then the required local unitary is

$$U_{\mathbf{r},a;\mathbf{r},b}^{\leftrightarrow} = \sum_{\alpha \neq \beta} |\mathbf{r}, a, \beta\rangle \otimes |\mathbf{r}, b, \alpha\rangle \langle \mathbf{r}, a, \alpha| \otimes \langle \mathbf{r}, b, \beta| + \delta_{\alpha\beta} \mathbb{I}_{\mathbf{r},a;\mathbf{r},b}. \quad (\text{S10})$$

Local unitaries of this form from each site commute, and so the complete product may be generated by a local Hamiltonian acting for a finite time. The action of this local unitary changes the effect of the drive to a product of pairwise swaps of states between neighbouring sites, as shown in Fig. S2(b).

Once in this form, it is clear that the action of the unitary is local and that it belongs to the trivial permutation class. Explicitly, the transformed unitary may now be written

$$U^{L(1,k) \otimes R(1,k)} = \bigotimes_{\mathbf{r}} U_{\mathbf{r},b;\mathbf{r}+\hat{\mathbf{x}},a}^{\leftrightarrow}, \quad (\text{S11})$$

where $U_{\mathbf{r},b;\mathbf{r}+\hat{\mathbf{x}},a}^{\leftrightarrow}$ is the local pairwise exchange between the state in Hilbert space b on site \mathbf{r} , and the state in Hilbert space a on site $\mathbf{r} + \hat{\mathbf{x}}$. Its action may be reduced to the identity by the inverse operation, which is a local, finite-time unitary.

Thus, we see that $L(1,k) \otimes R(1,k)$ is trivial, and it follows from this relation that

$$L(1, nk') \otimes R(1, nk) \equiv L(1, k') \otimes R(1, k). \quad (\text{S12})$$

By gathering together left- and right-moving parts of the permutation, and then cancelling common integer factors, a general composition of drives may be written $L(p', k') \otimes R(p, k) \equiv L(1, (k')^{p'}) \otimes R(1, k^p)$.